Approximate Solution of Schrodinger Equation in D-Dimensions for Scarf Hyperbolic Potential Using Nikiforov-Uvarov Method

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Abstract – The approximate analytical solution of Schrodinger equation in D-Dimensions for Scarf hyperbolic potential were investigated using Nikiforov-Uvarov method. The approximate bound state energy are given in the close form and the corresponding approximate wave function for arbitrary l-state in D-dimensions are formulated in the form of generalized Jacobi Polynomials. Special case is given for 5, 6, and 7 dimension from ground state to third excited state of bound state energy and wave function. The effect of the presence of this potential decreases the energy spectrum of Scarf hyperbolic potential.

Key words: Schrodinger equation in D-dimensions, Scarf hyperbolic potential, Nikiforov-Uvarov method, bound state energy, and wave function.

I. INTRODUCTION
The analytical solutions of Schrodinger equations for some physical potentials are very essential since they provide the important information of the quantum system. Recently, considerable efforts have been paid to obtain the exact solution of the shape invariant potentials. These potentials include Coulomb, Morse, Poschl-Teller, Hulten, inverted generalized hyperbolic potentials. The bound state energy spectra of these potentials have been investigated by various techniques such as Coulomb potential using Laplace transformation [1], Morse potential using series expansion method [2], Modified Poschl-Teller and Hulten potential using NU method [3,4]. However, some shape invariant potentials in D-dimensions still unresolved. The exact solution of Schrodinger equation is obtained if the angular momentum \( l = 0 \) in one dimension. Nevertheless, for \( l \neq 0 \) and \( D > 1 \), the Schrodinger equation can only be solved approximately for different suitable approximation scheme. One of the suitable approximation scheme is conventionally proposed by Greene and Aldrich [5,6].

In this paper we will attempt to solve the Schrodinger equation in D-Dimensions for Scarf potential using Nikiforov Uvarov method. The NU method which was developed by Nikiforov-Uvarov [7]. This method is based on solving the second order linear differential equation by reducing it to a generalized equation of hypergeometric type by a suitable change of variable.

One of the shape invariant potential in D-dimensions that remain unresolved is Scarf potential. This potentials describe particles which periodically arranged such as a crystal [8]. The application of solution in this potential are crystal model in solid state physics [9]. This potential can be transformed into a hyperbolic form, that is, Scarf hyperbolic potential. The solution this potential still remain in one dimension [10] and three dimensions [11]. In D-dimensions, this potential with the centrifugal term is separable potential, therefore, it can be solved using variable separation method, and has potential application in describing a structure of crystal for \( l \neq 0 \) in higher dimensions.

This paper is organized as follows. In section 2, we review the Nikiforov-Uvarov (NU) method briefly. In section 3, we find the bound state energy solution in term of new orbital number and wave functions. In section 4, we give some example of bound state energy and wave function for ground state to third excited state in 5, 6, and 7 dimension. A brief results and conclusion in section five.

II. REVIEW OF NIKIFOROV-UVAROV METHOD
The one-dimensional Schrodinger equation of any shape invariant potential can be reduced into hypergeometric or confluent hypergeometric type differential equation by suitable variable transformation \([12-14]\). The hypergeometric type differential equation, which is solved using Nikiforov-Uvarov method, is presented as

\[
\frac{d^2}{dx^2} \psi(x) + \frac{\sigma(x)}{x} \frac{d}{dx} \psi(x) + \frac{\tau(x)}{x^2} \psi(x) = 0
\]

(1)

where \( \sigma(x) \) and \( \tau(x) \) are polynomials at most in the second order, and \( \psi(x) \) is first order polynomial. Equation (1) can be solved using separation of variable method which is expressed as

\[
\psi = \phi(x) \chi(x)
\]

(2)

By inserting equation (2) into equation (1) we get hypergeometric type equation

\[
\sigma \phi'' + \tau \phi' + \lambda \phi = 0
\]

(3)

and \( \phi' \) is a logarithmic derivative whose solution obtained from condition

\[
\phi' = \frac{\pi}{\sigma}
\]

(4)

while the function \( \pi(x) \) and the parameter \( \lambda' \) are defined as

\[
\pi = \left( \frac{\tau - \sigma}{\lambda} \right) \pm \sqrt{\left( \frac{\tau - \sigma}{\lambda} \right)^2 - \sigma + k \sigma}
\]

(5)

\[
\lambda = k + \pi'
\]

(6)

The value of \( k \) in equation (5) can be found from the condition that the expression under the square root of
quadratic expression is zero. A new eigenvalue of equation (3) is
\[ \lambda = \lambda_n = -n\pi^2 - \frac{n(n-1)}{2}\alpha^2 \], \( n = 0, 1, 2 \) (7)

where \( \alpha = \frac{r}{r} \). The new energy eigenvalue is obtained using equation (6) and (7).

To generate the energy eigenvalues and the corresponding eigenfunctions, the condition that \( \alpha' < 0 \) is required. The solution of the second part of the wave function, \( y_n(z) \), which is connected to Rodrigues relation [15], is given as
\[ y_n(z) = \frac{c_n}{\rho(z)} \frac{d^n}{d\rho^n} \left( \rho^n \beta(z) \right) \] (9)

where \( c_n \) is normalization constant, and the weight function \( \rho(z) \) must satisfies the condition
\[ \frac{d\rho}{dz} = \alpha(z) \rho(z) \] (10)

The wave function of the system is therefore obtained from equation (4), (9) and (10).

### III. SCHRODINGER EQUATION IN D-DIMENSIONS FOR SCARF HYPERBOLIC POTENTIAL

The D-dimensional Schrodinger equation for Scarf hyperbolic potential is expressed as
\[ -\frac{\hbar^2}{2m} \nabla^2 \psi(r, \Omega) + \frac{\hbar^2}{2m} \left( b^2 + a(a + 1) \frac{\cosh r}{\sinh^2 r} \right) \psi(r, \Omega) = E\psi(r, \Omega) \] (11)

Where \( \nabla^2 \) is laplacian in D-dimensions, that is
\[ \nabla^2 = \frac{1}{r^{d-1}} \frac{\partial}{\partial r} \left( r^{d-1} \frac{\partial}{\partial r} \right) - \Lambda_n^2(\Omega_0) \] (12)

And \( \Lambda_n^2(\Omega_0) \) are the hyperspherical harmonics with
\[ \Lambda_n^2(\Omega_0) = \sum_{i=0}^{D-2} \left( \prod_{i} \sin \theta_i \right)^{D-2} \left( \sin \theta_1 \right)^{i+D} \frac{\partial}{\partial \theta_1} \left( \sin \theta_2 \right)^{D-1} \frac{\partial}{\partial \theta_2} + \left( \prod_{i} \sin \theta_i \right)^{D-2} \frac{\partial^2}{\partial \phi^2} \] (13)

Since the non-central potential is separable one, then equation (11) is solved using variable separation method by setting the wave function in equation (11) as \( \psi(r, \Omega) = \Psi(r)Y(\Omega) \) and we obtain:
\[ r^{-1} \frac{\partial}{\partial r} \left( r^{D-1} \frac{\partial \Psi(r)}{\partial r} \right) - r^2 \left( b^2 + a(a + 1) \frac{\cosh r}{\sinh^2 r} - 2b(a + \frac{1}{2}) \cosh r + \epsilon^2 \right) \psi(r) = \frac{1}{Y(\Omega)} \Lambda_n^2(\Omega_0) Y(\Omega) = \Lambda' = l(l+1) \] (14)

Where
\[ \epsilon^2 = \frac{2m}{\hbar^2} E \] (15)

For angular part, we have
\[ \Lambda_n^2(\Omega_0) Y(\Omega) - \Lambda \psi(\Omega) = 0 \] (16)

For radial part, we have
\[ \frac{d}{dr} \left( r^{D-1} \frac{d}{dr} \psi(r) \right) = \left( b^2 + a(a + 1) \frac{\cosh r}{\sinh^2 r} - 2b(a + \frac{1}{2}) \cosh r + \epsilon^2 + \frac{\Lambda'}{r} \right) r^{D-1} \psi(r) = 0 \] (17)

### A. Bound State Energy of Schrodinger Equation in D-dimensions for Scarf Hyperbolic Potential

The radial part of Schrodinger equation in D-dimensions for Scarf hyperbolic potential (17) can be solved using substitution of variable
\[ \Psi(r) = r^{-k} R_n(r) \] (18)

and approximation, for small \( r \), we have:
\[ r = \sinh r \] (19)

Substitute (18) and (19) to (17), we obtain
\[ \frac{d^2}{dr^2} R_n(r) - \left( \frac{b^2 + a(a+1) - 2b(a+\frac{1}{2}) \cosh r + \varepsilon^2 \sinh^2 r + \lambda' - \frac{(D-1)(D-3)}{4}}{\sinh^2 r} \right) R_n(r) = 0 \]  

Equation (20) can be expressed to (1) with substitution of variable:
\[ \cosh r = s \]  

We have
\[ \frac{d^2}{ds^2} R_n(s) + \frac{s}{(s^2 - 1)} \frac{d}{ds} R_n(s) - \left[ \frac{b^2 + a(a+1) - 2b(a+\frac{1}{2}) s + \varepsilon^2 (s^2 - 1) + \lambda' - \frac{(D-1)(D-3)}{4}}{(s^2 - 1)^2} \right] R_n(s) = 0 \]  

By comparing equation (1) and (22), we have
\[ \tau = s \]  
\[ \sigma = (s^2 - 1) \]  

\[ \tilde{\sigma} = -\left[ b^2 + a(a+1) - 2b(a+\frac{1}{2}) s + \varepsilon^2 (s^2 - 1) + \lambda' - \frac{(D-1)(D-3)}{4} \right] \]  

(23c)

From equation (5) and (23), we have
\[ \pi = \frac{s}{2} \pm \frac{1}{2} \sqrt{\left[ \frac{4}{s^2} + \varepsilon^2 + k \right] s^2 - 2b(a+\frac{1}{2}) s + \left[ b^2 + a(a+\frac{1}{2})^2 + \lambda' - \frac{(D-1)(D-3)}{4} \right] - \frac{1}{4} - \varepsilon^2 - k} \]  

(24)

The value of k is obtained from the condition that quadratic expression under the square root in equation (24) has to be square of first degree of polynomial therefore equation (24) is rewritten as
\[ \pi = \frac{s}{2} \pm p \left( s - \frac{2b(a+\frac{1}{2})}{2p^2} \right) \]  

(25)

With
\[ p^2 = \frac{1}{4} + \varepsilon^2 + k \]  

and the discriminate of the quadratic expression under the square root that has to be zero is given as
\[ b^2 \left( a + \frac{1}{2} \right)^2 - p^2 \left( b^2 + \left( a + \frac{1}{2} \right)^2 + \lambda' - \frac{(D-1)(D-3)}{4} - p \right) = 0 \]  

(27)

From equation (27), we obtain the value of p as
\[ p_1^2 = \frac{\left[ b^2 + (a+\frac{1}{2})^2 + \lambda' - \frac{(D-1)(D-3)}{4} \right] + \left[ \frac{(b+(a+\frac{1}{2}))^2 + \lambda' - \frac{(D-1)(D-3)}{4}}{\varepsilon^2 - \frac{1}{4}} \right] + \left[ \frac{(b+(a+\frac{1}{2}))^2 + \lambda' - \frac{(D-1)(D-3)}{4}}{\varepsilon^2 - \frac{1}{4}} \right]}{2} \]  

(28a)

\[ p_2^2 = \frac{\left[ b^2 + (a+\frac{1}{2})^2 + \lambda' - \frac{(D-1)(D-3)}{4} \right] - \left[ \frac{(b+(a+\frac{1}{2}))^2 + \lambda' - \frac{(D-1)(D-3)}{4}}{\varepsilon^2 - \frac{1}{4}} \right] - \left[ \frac{(b+(a+\frac{1}{2}))^2 + \lambda' - \frac{(D-1)(D-3)}{4}}{\varepsilon^2 - \frac{1}{4}} \right]}{2} \]  

(28b)

To find the correct parameter (p1 or p2), we reduce equation (28) for special condition which have one Dimension (D=1) and the sentrifugal factor vanished (\( \lambda' = 0 \)). Thus, we have
\[ p_1^2 = b^2 \]  

(29a)
\[ p_2^2 = \left( a + \frac{1}{2} \right)^2 \]  

(29b)

and
\[ k_1 = b^2 - \frac{1}{4} - \varepsilon^2 \]  

(30a)
\[ k_2 = \left( a + \frac{1}{2} \right)^2 - \frac{1}{4} - \varepsilon^2 \]  

(30b)

Parameter \( \pi_1 \) and \( \pi_2 \) for special case obtained from equation (25), there are
\[ \lambda_n = -2n(1-b) - n(n-1) \]  \hspace{1cm} (34a)
\[ \lambda_n = -n(2a) - n(n-1) \]  \hspace{1cm} (34b)

For first parameter \( \lambda = \lambda_n \), we have the bound state energy
\[ E_n = -\frac{\hbar^2}{2m} \left( b - n \right)^2 + n + b - \frac{1}{2} \]  \hspace{1cm} (35a)

Comparing the result of energy eigenvalue for special case (Equation 35) with the Suparmi’s earlier research (Equation 36), the parameter which have a physical meaning are the second parameter of \( p \), for general condition (in D-dimensions with 
\[ \text{sentrifugal term}, \] there are:
\[ \rho^2 = 2 \left\{ b + (a + \frac{1}{2}) \right\}^2 + \chi' - \frac{(D-3)(D-1)}{4} \]
\[ k = 2 \left\{ b + (a + \frac{1}{2}) \right\}^2 + \chi' - \frac{(D-3)(D-1)}{4} \]

Ansatz, for general case, we have
\[ \tau = 2 \left\{ s - ps + \frac{b(a + \frac{1}{2})}{p} \right\} \]
\[ \pi = \frac{1}{2} s - ps + \frac{b(a + \frac{1}{2})}{p} \]
\[ \lambda = \frac{1}{2} + p^2 - p - \epsilon^2 \]
\[ \lambda_n = -n^2 - n + 2np \]

\[ \text{The energy eigenvalue can be obtained in condition that } \lambda = \lambda_n, \text{ from (40) and (41), that is} \]
\[ E_n = -\frac{\hbar^2}{2m} (n - p + \frac{1}{2})^2 \]  \hspace{1cm} (42)

B. Wave function of Schrodinger Equation in D-dimensions for Scarf hyperbolic potential

The first part of wave function can be obtained from equation (4), that is
\[ \phi = \left( s^2 - 1 \right)^{\frac{1}{2}p} (s + 1)^{-\frac{1}{2}(a+\frac{1}{2})} (s - 1)^{\frac{1}{2}(a+\frac{1}{2})} \]
\[ \rho(s) = \left( s^2 - 1 \right)^r (S + 1)^{\frac{\mu(a+\frac{1}{2})}{p}} (S - 1)^{\frac{\mu(a+\frac{1}{2})}{p}} \]

or can be expressed in Jacobi’s Polynomial, that is
\[ y_n = 2^n n! P_n^{(a, b)}(s) = B_n P_n^{(a, b)}(s) \]

With
\[ P_n^{(a, b)}(s) = \frac{(-1)^n}{2^n n!} \left( s \right)^{a} (s + 1)^{b} \left( s - 1 \right)^{a} \left( s + 1 \right)^{b} \left( s - 1 \right)^{n} \]  \hspace{1cm} (47)
\[ \alpha = \frac{b(a + \frac{1}{2})}{p} - p \]  
\[ \beta = \frac{-b(a + \frac{1}{2})}{p} - p \]  

(48a)

(48b)

The wave function obtained from equation (2), that is

\[ R_n(s) = B_n \left( s^2 - 1 \right)^{\frac{1}{2} + \beta} \left( s + 1 \right)^{\frac{1}{2} + \alpha} \left( s - 1 \right)^{\frac{1}{2} + (\alpha + 1)} P_n^{(\alpha, \beta)}(s) \]  

(49)

Where, \( B_n \) is a normalization constant. From the normalization of wave function, we have

\[ B_n = \left( -1 \right)^{\frac{d-1}{2} + p} \sqrt{\frac{2n + \alpha + \beta + 1}{2^\alpha \beta + 1}} \frac{\Gamma(n + \alpha + \beta + 1)n!}{\Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)} \]  

(50)

Finally, from equation (18), (19), and (49), we have the radial wave function of Schrodinger equation for Scarf hyperbolic potential, that is

\[ \Psi_n(r) = B_n r^{\frac{d-1}{2} - p} \left( \sinh r \right)^{\frac{1}{2} + \beta} \left( \cosh r + 1 \right)^{\frac{1}{2} + \alpha} \left( \cosh r - 1 \right)^{\frac{1}{2} + (\alpha + 1)} P_n^{(\alpha, \beta)}(\cosh r) \]  

(51)

IV. EXAMPLE FOR \( \Psi_1(r) \), \( \Psi_2(r) \), AND \( \Psi_3(r) \) IN 5, 6, AND 7 DIMENSIONS

From equation (47), we have the value \( P_1, P_2, \) and \( P_3 \), that is

\[ P_n^{(\alpha, \beta)}(s) = 1 \]

(52a)

\[ P_n^{(\alpha, \beta)}(s) = \frac{1}{2} \left[ (1 - s^2) \left\{ (\alpha - 1)^2 - \beta (1 + s)^2 \right\} + 2s \right] \]

(52b)

\[ P_n^{(\alpha, \beta)}(s) = \frac{1}{8} \left[ (1 - s^2)^2 \left\{ (\alpha - 1)^2 (1 - s)^2 + \beta (\beta - 1)(1 + s)^2 \right\} \right. \]

\[ \left. -2 \left\{ 2\beta s (1 + s)^{-1} + 2\beta s (1 - s)^{-1} + \alpha \beta \right\} + 8s^2 \right] \]

(52c)

\[ P_n^{(\alpha, \beta)}(s) = \frac{1}{48} \left[ (1 - s^2)^2 \left\{ (1 - s)^{-1} \left( 3\alpha (\alpha - 1) + 6\alpha \right) + (1 + s)^{-1} \left( -\alpha \beta (\beta - 1) - 18\beta \right) \right\} \right. \]

\[ \left. + (1 - s)^2 \left\{ 36\alpha \beta s - 24\alpha s (1 - s)^{-1} (1 + 2s) + 24(1 + s)^{-1} (2 + \beta s) + 24 \right\} - 48 \right] \]

(52d)

From equation (28), (48), (51), and (42), we have

Table 1. Bound State Energy and Wave Function of Schrodinger Equation in 5, 6, and 7 Dimension for Scarf Hyperbolic Potential

<table>
<thead>
<tr>
<th>No</th>
<th>D Parameter</th>
<th>Wave Function</th>
<th>Bound state energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>[ \Psi_0(r) = \frac{B_n r^{-2} (\sinh r)^{\frac{1}{2} + \beta} \left( \cosh r + 1 \right)^{\frac{1}{2} + \alpha} \left( \cosh r - 1 \right)^{\frac{1}{2} + (\alpha + 1)} P_n^{(\alpha, \beta)}(\cosh r) ] ]</td>
<td>( E_0 = \frac{\hbar^2}{2m} \left( \frac{1}{2} - p \right)^2 )</td>
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<tr>
<td></td>
<td>5</td>
<td>[ \Psi_1(r) = \frac{B_n r^{-2} (\sinh r)^{\frac{1}{2} + \beta} \left( \cosh r + 1 \right)^{\frac{1}{2} + \alpha} \left( \cosh r - 1 \right)^{\frac{1}{2} + (\alpha + 1)} P_n^{(\alpha, \beta)}(\cosh r) ] ]</td>
<td>( E_1 = \frac{\hbar^2}{2m} \left( \frac{1}{2} - p \right)^2 )</td>
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<tr>
<td></td>
<td>5</td>
<td>[ \Psi_2(r) = \frac{B_n r^{-2} (\sinh r)^{\frac{1}{2} + \beta} \left( \cosh r + 1 \right)^{\frac{1}{2} + \alpha} \left( \cosh r - 1 \right)^{\frac{1}{2} + (\alpha + 1)} P_n^{(\alpha, \beta)}(\cosh r) ] ]</td>
<td>( E_2 = \frac{\hbar^2}{2m} \left( \frac{1}{2} - p \right)^2 )</td>
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<tr>
<td></td>
<td>6</td>
<td>[ \Psi_3(r) = \frac{B_n r^{-2} (\sinh r)^{\frac{1}{2} + \beta} \left( \cosh r + 1 \right)^{\frac{1}{2} + \alpha} \left( \cosh r - 1 \right)^{\frac{1}{2} + (\alpha + 1)} P_n^{(\alpha, \beta)}(\cosh r) ] ]</td>
<td>( E_3 = \frac{\hbar^2}{2m} \left( \frac{1}{2} - p \right)^2 )</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>[ \Psi_4(r) = \frac{B_n r^{-2} (\sinh r)^{\frac{1}{2} + \beta} \left( \cosh r + 1 \right)^{\frac{1}{2} + \alpha} \left( \cosh r - 1 \right)^{\frac{1}{2} + (\alpha + 1)} P_n^{(\alpha, \beta)}(\cosh r) ] ]</td>
<td>( E_4 = \frac{\hbar^2}{2m} \left( \frac{1}{2} - p \right)^2 )</td>
</tr>
</tbody>
</table>
\[ p^2 = \sum_{n=1}^{N} \left\{ b_n (a + \beta)^2 + \lambda_n - \beta_n \right\} \]

\[ p^2 = \sum_{n=1}^{N} \left\{ b_n (a + \beta)^2 + \lambda_n - \beta_n \right\} \]

\[ \alpha = \frac{b(a + \frac{1}{2})}{p} - \frac{p}{p} \]

\[ \beta = -\frac{b(a + \frac{1}{2})}{p} - \frac{p}{p} \]

**V. CONCLUSION**

In this paper, we present the solutions of Schrodinger equation in D-dimension for Scarf hyperbolic potential within the framework of an approximation to the centrifugal term. The bound state energy were obtained in D-dimensions using the Nikiforov-Uvarov method, and it was found to agree with previous works \(^{[10-11]}\). The effect of the presence of this potential decreases the energy spectrum of Scarf hyperbolic potential. The corresponding wave function of the Scarf hyperbolic potential were obtained in terms of the Jacobi polynomials. The example of bound state energy and wave function in 5, 6, and 7 dimension presented in condition of ground state to third excited state. Finally, the normalization constants were obtained in the form of gamma function.

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TANYA JAWAB

Muhammad Darwis Umar, Fisika UGM

Utama Alan Deta

√ Antara variabelnya independent satu dengan yang lain.

Apa syarat separasi variabel?